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Dirac–Schwinger Commutation Relations on a Lattice

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Low-frequency excitations of a one-dimensional chain of oscillators propagate like waves with a uniform velocity. Therefore this lattice is Lorentz invariant on a macroscopic scale. The Dirac–Schwinger commutation relations are constructed explicitly and shown to have the correct limit for long wavelengths. A similar test could be used to check the Lorentz invariance of lattice field theories.

It was shown long ago by Dirac (1962) and Schwinger (1962) that a simple criterion for Lorentz invariance in quantum field theory is the commutation relation

$$[\mathcal{H}(\mathbf{x}),\mathcal{H}(\mathbf{y})]/i\hbar = -[\mathcal{P}_k(\mathbf{x})+\mathcal{P}_k(\mathbf{y})]\partial^k\delta(\mathbf{x}-\mathbf{y}) \tag{1}$$

relating the energy density to the momentum density. Recently, there has been considerable interest in field theories on a lattice (Wilson, 1974; Kogut and Susskind, 1975; Drell et al., 1976; 1977; 1978). It is generally believed that Lorentz invariance can be achieved in the limit where the lattice parameter tends to zero. However, this is a difficult limit to evaluate and formal proofs of this assertion are extremely intricate, even for the simplest model field theories (Glimm and Jaffe, 1973; Park, 1975; Feldman and Osterwalder, 1976; Magnen and Seneor, 1976; McCoy and Wu, 1978). The purpose of this note is to show how the Dirac–Schwinger criterion can be applied to a lattice.

As an illustration, consider a one-dimensional lattice of nonrelativistic harmonic oscillators with nearest-neighbor interaction. It is well known (Goldstein, 1951) that low-frequency excitations of that lattice propagate like waves with uniform velocity. Therefore, in the limit of long wavelengths (many lattice parameters) the collective modes should display a kind of

Lorentz invariance, c being the speed of *sound*. The calculations are (almost) straightforward. With suitable units, the Hamiltonian is

$$H=\tfrac{1}{2}\Sigma\left[p_k^2+(q_{k+1}-q_k)^2\right] \tag{2}$$

and it is natural to define the "Hamiltonian density" as

$$H_k=\tfrac{1}{2}\left[p_k^2+\tfrac{1}{2}(q_{k+1}-q_k)^2+\tfrac{1}{2}(q_k-q_{k-1})^2\right] \tag{3}$$

We obtain

$$[H_k, H_m]/i\hbar=P_k(\delta_{k,m-1}-\delta_{km})+P_m(\delta_{km}-\delta_{k,m+1}) \tag{4}$$

where

$$P_k=\tfrac{1}{2}(q_{k+1}-q_k)(p_{k+1}+p_k) \tag{5}$$

is the "momentum density". [A more suggestive notation might be to call this $P_{k+1/2}$ and to express the rhs of (4) in terms of $P_{k\pm 1/2}$.] We can now define the total field momentum as

$$P=\Sigma P_k \tag{6}$$

and easily check that

$$[H, P]=0 \tag{7}$$

Likewise, it readily follows from equations (4) and (5) that the "boost" operator

$$K=\Sigma kH_k \tag{8}$$

satisfies

$$[H, K]/i\hbar=P \tag{9}$$

However, we also get, after a lengthy calculation

$$[P, K]/i\hbar=\tfrac{1}{2}\Sigma[p_k p_{k+1}+(q_{k+1}-q_k)(q_k-q_{k-1})] \tag{10}$$

rather than

$$[P, K]/i\hbar=H \tag{11}$$

as required for the Lorentz group. Of course, we should not be surprised that something went "wrong" because the lattice is *not* Lorentz invariant. The present calculation shows explicitly how Lorentz invariance is attained in the limit of long wavelengths, when arithmetic and geometric means are almost equal:

$$(q_{k+1}-q_k)(q_k-q_{k-1}) \simeq \tfrac{1}{2}\left[(q_{k+1}-q_k)^2+(q_k-q_{k-1})^2\right] \qquad (12)$$

and

$$\Sigma\, p_k p_{k+1} = \tfrac{1}{2}\Sigma\, p_k(p_{k+1}+p_{k-1}) \simeq \Sigma\, p_k^2 \qquad (13)$$

In conclusion, it should be noted that the Dirac–Schwinger relations, which guarantee that the fields transform *locally* under the Poincaré group, are a stronger requirement that the *global* equations (7), (9), and (11).

REFERENCES

Dirac, P. A. M. (1962). *Reviews of Modern Physics*, **34**, 592.
Drell, S. D., et al. (1976). *Physical Review D*, **14**, 487, 1627.
Drell, S. D., et al. (1977). *Physical Review D*, **16**, 1769.
Drell, S. D., et al. (1978). *Physical Review D*, **17**, 523.
Feldman, J. S., and Osterwalder, K. (1976). *Annals of Physics*, **97**, 80.
Glimm, J., and Jaffe, A. (1973). *Fortschritte der Physik*, **21**, 327.
Goldstein, H. (1951). *Classical Mechanics*, p. 349. Addison-Wesley, Cambridge, Massachusetts.
Kogut, J., and Susskind, L. (1975). *Physical Review D*, **11**, 395.
McCoy, B. M., and Wu, T. T. (1978). *Physical Review D*, **18**, 1243, 1253, and 1259.
Magnen, J., and Seneor, R. (1976). *Annales de l'Institut Henri Poincaré*, **24**, 95.
Park, Y. M. (1975). *Journal of Mathematical Physics*, **16**, 1065, 2183.
Schwinger, J. (1962). *Physical Review*, **127**, 324.
Wilson, K. G. (1974). *Physical Review D*, **10**, 2445.